18.175 Theory of Probability
Fall 2008

Section 26

Laws of Brownian motion at stopping times. Skorohod's imbedding.

Let W_t be the Brownian motion.

Theorem 63 *If τ is a stopping time such that $\mathbb{E}\tau < \infty$ then $\mathbb{E}W_\tau = 0$ and $\mathbb{E}W_\tau^2 = \mathbb{E}\tau$.*

Proof. Let us start with the case when a stopping time τ takes finite number of values

$$\tau \in \{t_1, \ldots, t_n\}.$$

If $\mathcal{F}_{t_j} = \sigma\{W_t; t \leq t_j\}$ then $(W_{t_j}, \mathcal{F}_{t_j})$ is a martingale since

$$\mathbb{E}(W_{t_j}|\mathcal{F}_{t_{j-1}}) = \mathbb{E}(W_{t_j} - W_{t_{j-1}} + W_{t_{j-1}}|\mathcal{F}_{t_{j-1}}) = W_{t_{j-1}}.$$

By optional stopping theorem for martingales, $\mathbb{E}W_\tau = \mathbb{E}W_{t_1} = 0$. Next, let us prove that $\mathbb{E}W_\tau^2 = \mathbb{E}\tau$ by induction on n. If $n = 1$ then $\tau = t_1$ and

$$\mathbb{E}W_\tau^2 = \mathbb{E}W_{t_1}^2 = t_1 = \mathbb{E}\tau.$$

To make an induction step from $n - 1$ to n, define a stopping time $\alpha = \tau \wedge t_{n-1}$ and write

$$\mathbb{E}W_\tau^2 = \mathbb{E}(W_\alpha + W_\tau - W_\alpha)^2 = \mathbb{E}W_\alpha^2 + \mathbb{E}(W_\tau - W_\alpha)^2 + 2\mathbb{E}W_\alpha(W_\tau - W_\alpha).$$

First of all, by induction assumption, $\mathbb{E}W_\alpha^2 = \mathbb{E}\alpha$. Moreover, $\tau \neq \alpha$ only if $\tau = t_n$ in which case $\alpha = t_{n-1}$. The event

$$\{\tau = t_n\} = \{\tau \leq t_{n-1}\}^c \in \mathcal{F}_{t_{n-1}}$$

and, therefore,

$$\mathbb{E}W_\alpha(W_\tau - W_\alpha) = \mathbb{E}W_{t_{n-1}}(W_{t_n} - W_{t_{n-1}})\mathrm{I}(\tau = t_n) = 0.$$

Similarly,

$$\mathbb{E}(W_\tau - W_\alpha)^2 = \mathbb{E}\mathbb{E}(\mathrm{I}(\tau = t_n)(W_{t_n} - W_{t_{n-1}})^2|\mathcal{F}_{t_{n-1}}) = (t_n - t_{n-1})\mathbb{P}(\tau = t_n).$$

Therefore,

$$\mathbb{E}W_\tau^2 = \mathbb{E}\alpha + (t_n - t_{n-1})\mathbb{P}(\tau = t_n) = \mathbb{E}\tau$$

and this finishes the proof of the induction step. Next, let us consider the case of a uniformly bounded stopping time $\tau \leq M < \infty$. In the previous lecture we defined a dyadic approximation

$$\tau_n = \frac{\lfloor 2^n \tau \rfloor + 1}{2^n}$$

which is also a stopping time, $\tau_n \downarrow \tau$, and by sample continuity $W_{\tau_n} \to W_\tau$ a.s. Since (τ_n) are uniformly bounded, $\mathbb{E}\tau_n \to \mathbb{E}\tau$. To prove that $\mathbb{E}W_{\tau_n}^2 \to \mathbb{E}W_\tau^2$ we need to show that the sequence $(W_{\tau_n}^2)$ is uniformly integrable. Notice that $\tau_n < 2M$ and, therefore, τ_n takes possible values of the type $k/2^n$ for $k \leq k_0 = \lfloor 2^n(2M) \rfloor$. Since the sequence

$$\left(W_{1/2^n}, \ldots, W_{k_0/2^n}, W_{2M}\right)$$

is a martingale, adapted to a corresponding sequence of $\mathcal{F}_t$, and τ_n and $2M$ are two stopping times such that $\tau_n < 2M$, by Optional Stopping Theorem 31, $W_{\tau_n} = \mathbb{E}(W_{2M}|\mathcal{F}_{\tau_n})$. By Jensen's inequality,

$$W_{\tau_n}^4 \leq \mathbb{E}(W_{2M}^4|\mathcal{F}_{\tau_n}), \ \ \mathbb{E}W_{\tau_n}^4 \leq \mathbb{E}W_{2M}^4 = 6M.$$

and uniform integrability follows by Hölder's and Chebyshev's inequalities,

$$\mathbb{E}W_{\tau_n}^2 \mathrm{I}(|W_{\tau_n}| > N) \leq (\mathbb{E}W_{\tau_n}^4)^{1/2}(\mathbb{P}(|W_{\tau_n}| > N))^{1/2} \leq \frac{6M}{N^2} \to 0$$

as $N \to \infty$, uniformly over n. This proves that $\mathbb{E}W_{\tau_n}^2 \to \mathbb{E}W_\tau^2$. Since τ_n takes finite number of values, by the previous case, $\mathbb{E}W_{\tau_n}^2 = \mathbb{E}\tau_n$ and letting $n \to \infty$ proves

$$\mathbb{E}W_\tau^2 = \mathbb{E}\tau. \tag{26.0.1}$$

Before we consider the general case, let us notice that for two bounded stopping times $\tau \leq \rho \leq M$ one can similarly show that

$$\mathbb{E}(W_\rho - W_\tau)W_\tau = 0. \tag{26.0.2}$$

Namely, one can approximate the stopping times by dyadic stopping times and using that by the optional stopping theorem $(W_{\tau_n}, \mathcal{F}_{\tau_n})$, $(W_{\rho_n}, \mathcal{F}_{\rho_n})$ is a martingale,

$$\mathbb{E}(W_{\rho_n} - W_{\tau_n})W_{\tau_n} = \mathbb{E}W_{\tau_n}\big(\mathbb{E}(W_{\rho_n}|\mathcal{F}_{\tau_n}) - W_{\tau_n}\big) = 0.$$

Finally, we consider the general case. Let us define $\tau(n) = \min(\tau, n)$. For $m \leq n, \tau(m) \leq \tau(n)$ and

$$\mathbb{E}(W_{\tau(n)} - W_{\tau(m)})^2 = \mathbb{E}W_{\tau(n)}^2 - \mathbb{E}W_{\tau(m)}^2 - 2\mathbb{E}W_{\tau(m)}(W_{\tau(n)} - W_{\tau(m)}) = \mathbb{E}\tau(n) - \mathbb{E}\tau(m)$$

using (26.0.1), (26.0.2) and the fact that $\tau(n), \tau(m)$ are bounded stopping times. Since $\tau(n) \uparrow \tau$, Fatou's lemma and the monotone convergence theorem imply

$$\mathbb{E}(W_\tau - W_{\tau(m)})^2 \leq \liminf_{n \to \infty}(\mathbb{E}\tau(n) - \mathbb{E}\tau(m)) = \mathbb{E}\tau - \mathbb{E}\tau(m).$$

Letting $m \to \infty$ shows that

$$\lim_{m \to \infty} \mathbb{E}(W_\tau - W_{\tau(m)})^2 = 0$$

which means that $\mathbb{E}W_{\tau(m)}^2 \to \mathbb{E}W_\tau^2$. Since $\mathbb{E}W_{\tau(m)}^2 = \mathbb{E}\tau(m)$ by the previous case and $\mathbb{E}\tau(m) \to \mathbb{E}\tau$ by the monotone convergence theorem, this implies that $\mathbb{E}W_\tau^2 = \mathbb{E}\tau$.

$\square$

Theorem 64 *(Skorohod's imbedding) Let Y be a random variable such that $\mathbb{E}Y = 0$ and $\mathbb{E}Y^2 < \infty$. There exists a stopping time $\tau < \infty$ such that $\mathcal{L}(W_\tau) = \mathcal{L}(Y)$.*

Proof. Let us start with the simplest case when Y takes only two values, $Y \in \{-a, b\}$ for $a, b > 0$. The condition $\mathbb{E}Y = 0$ determines the distribution of Y,

$$pb + (1 - p)(-a) = 0 \ \ \text{and} \ \ p = \frac{a}{a + b}. \tag{26.0.3}$$

Let $\tau = \inf\{t > 0, W_t = -a \text{ or } b\}$ be a hitting time of the two-sided boundary $-a, b$. The tail probability of τ can be bounded by

$$\mathbb{P}(\tau > n) \leq \mathbb{P}\big(|W_{j+1} - W_j| < a + b, 0 \leq j \leq n - 1\big) = \mathbb{P}(|W_1| < a + b)^n = \gamma^n.$$

Therefore, $\mathbb{E}\tau < \infty$ and by the previous theorem, $\mathbb{E}W_\tau = 0$. Since $W_\tau \in \{-a, b\}$ we must have

$$\mathcal{L}(W_\tau) = \mathcal{L}(Y).$$

Let us now consider the general case. If μ is the law of Y, let us define Y by the identity $Y = Y(x) = x$ on its sample probability space $(\mathbb{R}, \mathcal{B}, \mu)$. Let us construct a sequence of σ-algebras

$$\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots \subseteq \mathcal{B}$$

as follows. Let $\mathcal{B}_1$ be generated by the set $(-\infty, 0)$, i.e.

$$\mathcal{B}_1 = \{\emptyset, \mathbb{R}, (-\infty, 0), [0, +\infty)\}.$$

Given $\mathcal{B}_j$, let us define $\mathcal{B}_{j+1}$ by splitting each finite interval $[c, d) \in \mathcal{B}_j$ into two intervals $[c, (c+d)/2)$ and $[(c+d)/2, d)$ and splitting infinite interval $(-\infty, -j)$ into $(-\infty, -(j+1))$ and $[-(j+1), -j)$ and similarly splitting $[j, +\infty)$ into $[j, j+1)$ and $[j+1, \infty)$. Consider a right-closed martingale

$$Y_j = \mathbb{E}(Y | \mathcal{B}_j).$$

It is almost obvious that $\mathcal{B} = \sigma(\bigcup \mathcal{B}_j)$, which we leave as an exercise. Then, by the Levy martingale convergence, Lemma 35, $Y_j \to \mathbb{E}(Y | \mathcal{B}) = Y$ a.s. Since Y_j is measurable on $\mathcal{B}_j$, it must be constant on each simple set $[c, d) \in \mathcal{B}_j$. If $Y_j(x) = y$ for $x \in [c, d)$ then, since $Y_j = \mathbb{E}(Y | \mathcal{B}_j)$,

$$y\mu([c, d)) = \mathbb{E}Y_j I_{[c,d)} = \mathbb{E}Y I_{[c,d)} = \int_{[c,d)} x d\mu(x)$$

and

$$y = \frac{1}{\mu([c, d))} \int_{[c,d)} x d\mu(x). \tag{26.0.4}$$

Since in the σ-algebra $\mathcal{B}_{j+1}$ the interval $[c, d)$ is split into two intervals, the random variable Y_{j+1} can take only two values, say $y_1 < y_2$, on the interval $[c, d)$ and, since $(Y_j, \mathcal{B}_j)$ is a martingale,

$$\mathbb{E}(Y_{j+1} | \mathcal{B}_j) - Y_j = 0. \tag{26.0.5}$$

We will define stopping times τ_n such that $\mathcal{L}(W_{\tau_n}) = \mathcal{L}(Y_n)$ iteratively as follows. Since Y_1 takes only two values $-a$ and b, let $\tau_1 = \inf\{t > 0, W_t = -a \text{ or } b\}$ and we proved above that $\mathcal{L}(W_{\tau_1}) = \mathcal{L}(Y_1)$. Given τ_j define τ_{j+1} as follows:

$$\text{if } W_{\tau_j} = y \text{ for } y \text{ in } (26.0.4) \text{ then } \tau_{j+1} = \inf\{t > \tau_j, W_t = y_1 \text{ or } y_2\}.$$

Let us explain why $\mathcal{L}(W_{\tau_j}) = \mathcal{L}(Y_j)$. First of all, by construction, W_{τ_j} takes the same values as Y_j. If $\mathcal{C}_j$ is the σ-algebra generated by the disjoint sets $\{W_{\tau_j} = y\}$ for y as in $(26.0.4)$, i.e. for possible values of Y_j, then W_{τ_j} is $\mathcal{C}_j$ measurable, $\mathcal{C}_j \subseteq \mathcal{C}_{j+1}$, $\mathcal{C}_j \subseteq \mathcal{F}_{\tau_j}$ and at each step simple sets in $\mathcal{C}_j$ are split in two,

$$\{W_{\tau_j} = y\} = \{W_{\tau_{j+1}} = y_1\} \cup \{W_{\tau_{j+1}} = y_2\}.$$

By Markov's property of the Brownian motion and Theorem 63, $\mathbb{E}(W_{\tau_{j+1}} - W_{\tau_j} | \mathcal{F}_{\tau_j}) = 0$ and, therefore,

$$\mathbb{E}(W_{\tau_{j+1}} | \mathcal{C}_j) - W_{\tau_j} = 0.$$

Since on each simple set $\{W_{\tau_j} = y\}$ in $\mathcal{C}_j$, the random variable $W_{\tau_{j+1}}$ takes only two values y_1 and y_2, this equation allows us to compute the probabilities of these simple sets recursively as in $(26.0.3)$,

$$\mathbb{P}(W_{\tau_{j+1}} = y_2) = \frac{y_2 - y}{y_2 - y_1} \mathbb{P}(W_{\tau_j} = y).$$

By $(26.0.5)$, Y_j's satisfy the same recursive equations and this proves that $\mathcal{L}(W_{\tau_n}) = \mathcal{L}(Y_n)$. The sequence

τ_n is monotone, so it converges $\tau_n \uparrow \tau$ to some stopping time τ. Since

$$\mathbb{E}\tau_n = \mathbb{E}W_{\tau_n}^2 = \mathbb{E}Y_n^2 \leq \mathbb{E}Y^2 < \infty,$$

we have $\mathbb{E}\tau = \lim \mathbb{E}\tau_n \leq \mathbb{E}Y^2 < \infty$ and, therefore, $\tau < \infty$ a.s. Then $W_{\tau_n} \to W_\tau$ a.s. by sample continuity and since $\mathcal{L}(W_{\tau_n}) = \mathcal{L}(Y_n) \to \mathcal{L}(Y)$, this proves that $\mathcal{L}(W_\tau) = \mathcal{L}(Y)$.

$\square$